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On noncommutative and pseudo-Riemannian geometry

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We introduce the notion of a pseudo-Riemannian spectral triple which generalizes the notion of spectral triple and allows for a treatment of pseudo-Riemannian manifolds within a noncommutative setting. It turns out that the relevant spaces in noncommutative pseudo-Riemannian geometry are not Hilbert spaces any more but Krein spaces, and Dirac operators are Krein-selfadjoint. We show that the noncommutative tori can be endowed with a pseudo-Riemannian structure in this way. For the noncommutative tori as well as for pseudo-Riemannian spin manifolds the dimension, the signature of the metric, and the integral of a function can be recovered from the spectral data.

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1. Introduction

The Gel'fand–Naimark theorem states that any unital commutative C^*-algebra can be realized as an algebra of continuous functions on a compact Hausdorff space. In noncommutative geometry one thinks of a noncommutative C^*-algebra as an algebra of functions on some "virtual" space and tries to imitate geometrical constructions which work for the case of commutative algebras. Connes functional analytic approach (see [6]) to noncommutative geometry starts with the observation that the metric information of a compact Riemannian

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spin manifold M is encoded in the triple $(C^\infty(M), \not{D}, L^2(M, S))$, where $\not{D}$ is the Dirac operator and $L^2(M, S)$ is the Hilbert space of square integrable sections of the spinor bundle. The algebra $C^\infty(M)$ is realized as a $*$-algebra of bounded operators on $L^2(M, S)$. The space of characters of $C^\infty(M)$ is canonically isomorphic to the set of points of M and the Riemannian distance between to point p and q can be recovered from the equation

$$d(p, q) = \sup |f(p) - f(q)|, \quad f \in C^\infty(M), \ \|[\not{D}, f]\| \leq 1. \tag{1}$$

The noncommutative generalization of the object $(C^\infty(M), \not{D}, L^2(M, S))$ is the so called spectral triple, which we can think of as a generalization of the theory of compact Riemannian manifolds. For a further introduction to noncommutative geometry we would like to refer the reader to [9,16,22,17] and the references therein.

Recently there have been attempts to get analogues of spectral triples which allow for a treatment of non-compact manifolds (see [20]) and globally hyperbolic Lorentzian manifolds (see [18,15,14,13]). Such a treatment seems necessary if one wants to study physical models, which are defined on spaces with Lorentzian rather than Riemannian metrics. The idea in [10,15,14,13] is to foliate the space–time into Cauchy surfaces and to treat the Cauchy surfaces as Riemannian manifolds. Whereas this approach seems promising for the study of evolution equations in physics, its dependence on the foliation and the restriction to Lorentzian signatures is disturbing from the mathematical point of view.

In this paper we suggest a notion of pseudo-Riemannian spectral triple, which allows to treat compact pseudo-Riemannian manifolds (of arbitrary signature) within noncommutative geometry. Such a triple $(\mathcal{A}, D, \mathcal{H})$ consists of an involutive algebra $\mathcal{A}$ of bounded operators acting on a Krein space $\mathcal{H}$ and a Krein-selfadjoint operator D. An important role is played by the fundamental symmetries of the Krein space. These are operators $\mathfrak{J} : \mathcal{H} \to \mathcal{H}$ with $\mathfrak{J}^2 = 1$ such that $(\cdot, \mathfrak{J}\cdot) = (\mathfrak{J}\cdot, \cdot)$ is a positive definite scalar product turning $\mathcal{H}$ into a Hilbert space. They can be used to obtain ordinary spectral triples from pseudo-Riemannian spectral triples in a similar way as this is done in physics by "Wick rotation", which is used to pass to Riemannian signatures of the metric. For example if M is a Lorentzian spin manifold, $\mathcal{A} = C_0^\infty(M)$ and D is the Dirac operator which acts on the Krein space of square integrable sections $\mathcal{H}$ of the spinor bundle, the triple $(\mathcal{A}, D, \mathcal{H})$ is a pseudo-Riemannian spectral triple. From the pseudo-Riemannian metric g on M one can obtain a Riemannian metric by applying a so called spacelike reflection r. This is an involutive endomorphism $r : TM \to TM$ of vector bundles such that $g(r\cdot, \cdot)$ is a Riemannian metric. Such spacelike reflections give rise to a large class of fundamental symmetries $\mathfrak{J}$ such that the operator $(1/2)((\mathfrak{J}D)^2 + (D\mathfrak{J})^2)$ is a Laplace-type operator with respect to a Riemannian metric. We can think of this metric as a Wick rotated form of the Lorentzian metric. We use this to show that one can define a notion of dimension for pseudo-Riemannian spectral triples. In the commutative case and for the noncommutative pseudo-Riemannian torus we show that there is a canonical notion of integration and one can recover the signature of the metric from the spectral data. It should be noted that the "Wick rotation" takes place in the fibres of TM and therefore does not require a special choice of time coordinates.

In Sections 2–5 we review the basic notions and results on spectral triples, Krein spaces and Dirac operators on pseudo-Riemannian manifolds. Sections 6 and 7 contain the main results of this paper.

2. Spectral triples

Definition 2.1. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a unital $*$-algebra $\mathcal{A}$ of bounded operators on a separable Hilbert space $\mathcal{H}$ and a selfadjoint operator D on $\mathcal{H}$, such that the commutator $[D, a]$ is bounded for all $a \in \mathcal{A}$. A spectral triple is said to be even if there exists an operator $\chi = \chi^*$, $\chi^2 = 1$ on the Hilbert space such that

$$\chi a = a\chi \quad \forall a \in \mathcal{A}, \tag{2}$$

$$D\chi = -\chi D. \tag{3}$$

For a compact operator a denote by $\mu_k(a)$ the ordered sequence of its singular values, i.e. $\mu_k(a)$ are the eigenvalues of $|a|$ such that $\mu_1(a) \geq \mu_2(a) \geq \cdots$, with each eigenvalue repeated according to its multiplicity. The characteristic sequence of a is defined by $\sigma_k(a) := \sum_{i=1}^{k} \mu_i(a)$. Let $p \geq 1$ be a real number. A compact operator a is said to be in $\mathcal{L}^{p+}$ if

$$\sup_{n\geq 1} \frac{\sigma_n(a)}{n^{(p-1)/p}} < \infty \quad \text{for} \quad p > 1, \tag{4}$$

$$\sup_{n>2} \frac{\sigma_n(a)}{\ln n} < \infty \quad \text{for} \quad p = 1. \tag{5}$$

The spaces $\mathcal{L}^{p+}$ are two 2-sided ideals in $\mathcal{B}(\mathcal{H})$. Note that if $a \in \mathcal{L}^{p+}$, then $|a|^p \in \mathcal{L}^{1+}$.

Let now $l^\infty(\mathbb{N})$ be the von Neumann algebra of bounded sequences. If a state ω on $l^\infty(\mathbb{N})$ satisfies the conditions

- $\lim_{n\to\infty} x_n = x \Rightarrow \omega(x_n) = x$,
- $\omega(x_{2n}) = \omega(x_n)$,

we say that ω is in $\Gamma_s(l^\infty)$. The set $\Gamma_s(l^\infty)$ turns out to be non-empty [8]. For each positive $a \in \mathcal{L}^{1+}$ and each state $\omega \in \Gamma_s(l^\infty)$ we define $\mathrm{Tr}_\omega(a) := \omega(x(a))$, where $x(a)_n = \sigma_n(a)/\ln n$ for $n \geq 2$ and $x(a)_1 = 0$. It can be shown that for each ω the map $a \to \mathrm{Tr}_\omega(a)$ extends to a finite trace on $\mathcal{L}^{1+}$ and to a singular trace on $\mathcal{B}(\mathcal{H})$ (see [8,6]).

Definition 2.2. Let $p \geq 1$ be a real number. A spectral triple is called p^+-summable if $(1 + D^2)^{-1/2}$ is in $\mathcal{L}^{p+}$.

In case a spectral triple is p^+-summable the map $a \to \mathrm{Tr}_\omega(a(1 + D^2)^{-p/2})$ is well defined on the algebra $\mathcal{A}_D$ generated by $\mathcal{A}$ and $[D, \mathcal{A}]$. It can be shown that if $\mathcal{A}_D$ is contained in the domain of smoothness of the derivation $\delta(\cdot) := [|D|, \cdot]$, this map is a trace (see [5]). A differential operator on a Riemannian manifold is said to be of Dirac type if it is of first order and the principal symbol σ of D satisfies the relation

$$\sigma(\xi)^2 = g(\xi, \xi)\,\mathrm{id}_{E_p} \quad \forall \xi \in T_p^*M, \quad p \in M. \tag{6}$$

The geometry of a compact Riemannian spin manifold can be encoded in a spectral triple (see [6]).

Theorem 2.3 (Connes). *Let M be a compact Riemannian manifold of dimension n and E a hermitian vector bundle over M of rank k. Let $\mathcal{H}$ be the Hilbert space of square integrable sections of E and let $\mathcal{A}$ be $C^\infty(M)$ which acts on $\mathcal{H}$ by multiplication. Assume that D is a selfadjoint differential operator of Dirac type on E. Then $(\mathcal{A}, \mathcal{H}, D)$ is an n^+-summable spectral triple. As a compact space M is the spectrum of the C^*-algebra, which is the norm closure of $\mathcal{A}$. The geodesic distance on M is given by*

$$d(p, q) = \sup |f(p) - f(q)|, \quad f \in \mathcal{A}, \ \|[D, f]\| \leq 1. \tag{7}$$

Furthermore for $f \in C^\infty(M)$ we have

$$\int_M f\sqrt{|g|}\,\mathrm{d}x = c(n, k)\,\mathrm{Tr}_\omega(f(1 + |D|^2)^{-n/2}), \tag{8}$$

where $c(n, k) = 2^{n-1}\pi^{n/2}k^{-1}n\Gamma(n/2)$.

3. Differential calculus and spectral triples

Let $\mathcal{A}$ be a unital algebra. Denote by $\bar{\mathcal{A}}$ the vector space $\mathcal{A}/(\mathbb{C}1)$ and define $\Omega^n\mathcal{A} := \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes n}$. We write $(a_0, a_1, \ldots, a_n)$ for the image of $a_0 \otimes \cdots \otimes a_n$ in $\Omega^n\mathcal{A}$. On $\Omega\mathcal{A} := \oplus_{n=0}^\infty \Omega^n\mathcal{A}$ one now defines an operator d of degree 1 and a product by

$$d(a_0, \ldots, a_n) = (1, a_0, \ldots, a_n), \tag{9}$$

$$(a_0, \ldots, a_n)(a_{n+1}, \ldots, a_k) = \sum_{i=0}^{n}(-1)^{n-i}(a_0, \ldots, a_i a_{i+1}, \ldots, a_k). \tag{10}$$

This determines a differential algebra structure on $\Omega\mathcal{A}$. If $\mathcal{A}$ is a star algebra, one makes $\Omega\mathcal{A}$ a star algebra by $(a_0, \ldots, a_n)^* := (-1)^n(a_n^*, \ldots, a_1^*) \cdot a_0^*$. The pair $(\Omega\mathcal{A}, d)$ is called the universal differential envelope of $\mathcal{A}$.

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ gives rise to a $*$-representation of $\Omega\mathcal{A}$ on $\mathcal{H}$ by the map

$$\pi : \Omega\mathcal{A} \to \mathcal{B}(\mathcal{H}), \qquad \pi(a_0, a_1, \ldots, a_n) := a_0[D, a_1]\cdots[D, a_n], \quad a_j \in \mathcal{A}.$$

Let j_0 be the graded two-sided ideal $j_0 := \oplus_n j_0^n$ given by

$$j_0^n := \{\omega \in \Omega^n\mathcal{A}; \pi(\omega) = 0\}. \tag{11}$$

In general, j_0 is not a differential ideal. That is why it is not possible to define the space of forms to be the image $\pi(\Omega\mathcal{A})$. However $j := j_0 + dj_0$ is a graded differential two-sided ideal.

Definition 3.1. The graded differential algebra of Connes forms over $\mathcal{A}$ is defined by

$$\Omega_D\mathcal{A} := \frac{\Omega\mathcal{A}}{j} \cong \bigoplus_n \frac{\pi(\Omega^n\mathcal{A})}{\pi}(dj_0 \cap \Omega^n\mathcal{A}). \tag{12}$$

Example 3.2. The space of one-forms $\Omega^1_D\mathcal{A} \cong \pi(\Omega^1\mathcal{A})$ is the space of bounded operators of the form

$$\omega_1 = \sum_k a_0^k[D, a_1^k], \quad a_i^k \in \mathcal{A}. \tag{13}$$

Proposition 3.3. *Let $(\mathcal{A}, \mathcal{H}, D)$ be as in Theorem 2.3. As graded differential algebras $\Omega_D\mathcal{A}$ and $\Gamma(\Lambda M)$ are isomorphic.*

See [6, p. 552] or [16, Section 7.2.1] for a proof.

If the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is n^+-summable the map

$$w_1 \times w_2 \to \langle w_1, w_2\rangle := \mathrm{Tr}_\omega(w_1^* w_2(1 + |D|^2)^{-n/2}) \tag{14}$$

defines for each ω a scalar product on the space of one-forms. In the case of Proposition 3.3 this scalar product coincides up to a scalar factor with the metric-induced scalar product on the space of one-forms.

4. Krein spaces

4.1. Fundamentals

Let V be a vector space over $\mathbb{C}$. An indefinite inner product on V is a map $(\cdot, \cdot) : V \times V \to \mathbb{C}$ which satisfies

$$(v, \lambda w_1 + \mu w_2) = \lambda(v, w_1) + \mu(v, w_2), \qquad \overline{(v_1, v_2)} = (v_2, v_1).$$

The indefinite inner product is said to be non-degenerate, if

$$(v, w) = 0 \quad \forall v \in V \Rightarrow w = 0.$$

A non-degenerated indefinite inner product space V is called decomposable if it can be written as the direct sum of orthogonal subspaces V^+ and V^- such that the inner product is positive definite on V^+ and negative definite on V^-. The inner product then defines a norm on these subspaces. V^+ and V^- are called intrinsically complete if they are complete in these norms. A non-degenerate indefinite inner product space which is decomposable such that the subspaces V^+ and V^- are intrinsically complete is called a **Krein space**. For every decomposition $V = V^+ \oplus V^-$ the operator $\mathfrak{J} = \mathrm{id} \oplus -\mathrm{id}$ defines a positive definite inner product (the $\mathfrak{J}$-inner product) by $\langle\cdot, \cdot\rangle_{\mathfrak{J}} := (\cdot, \mathfrak{J}\cdot)$. Such an operator $\mathfrak{J}$ is called a fundamental symmetry. It turns out that if V is a Krein space each fundamental symmetry makes V a Hilbert space. Furthermore two Hilbert space norms associated to different fundamental symmetries are equivalent. The topology induced by these norms is called the strong topology on V. The theory of Krein spaces can be found in [4]. For the sake of completeness we will review in the following the main properties of linear operators on Krein spaces.

4.2. Operators on Krein spaces

If A is a linear operator on a Krein space V we say that A is densely defined if the domain of definition $\mathcal{D}(A)$ of A is strongly dense in V. Let A be a densely defined operator on a Krein space V. We may define the Krein adjoint A^+ in the following way. Let $\mathcal{D}(A^+)$ be the set of vectors v, such that there is a vector v^+ with

$$(v, Aw) = (v^+, w) \quad \forall w \in \mathcal{D}(A). \tag{15}$$

We set $A^+v := v^+$. A densely defined operator is called Krein-selfadjoint if $A = A^+$. An operator is called closed if its graph is closed in the strong topology, i.e. if the operator is closed as an operator on the Hilbert space associated to one (and hence to all) of the fundamental symmetries. If the closure of the graph of an operator A in the strong topology is an operator graph, then A is called closeable, the closure $\bar{A}$ is the operator associated with the closure of the operator graph. It turns out that a densely defined operator A is closeable if and only if A^+ is densely defined. The closure of A is then given by $\bar{A} = A^{++} := (A^+)^+$. Clearly, a Krein-selfadjoint operator is always closed. A densely defined operator is called essentially Krein-selfadjoint if it is closable and its closure is Krein-selfadjoint. Note that for any fundamental symmetry we have the equality $A^+ = \mathfrak{J}A^*\mathfrak{J}$, if the star denotes the adjoint in the Hilbert space defined by the $\mathfrak{J}$-inner product. Therefore, given a fundamental symmetry $\mathfrak{J}$ and a Krein-selfadjoint operator A, the operators $\mathfrak{J}A$ and $A\mathfrak{J}$ are selfadjoint as operators in the Hilbert space induced by the $\mathfrak{J}$-inner product. The symmetric operators $\mathrm{Re}((1/2)(A + A^*))$ and $\mathrm{Re}((\mathrm{i}/2)(A - A^*))$ are called real and imaginary parts of A. The sum of the squares of these operators is formally given by $(A)_{\mathfrak{J}} := (1/2)(A^*A + AA^*)$. It is natural to define the $\mathfrak{J}$-modulus of A as its square root. For a special class of fundamental symmetries this can be done straightforwardly. We have the following proposition.

Proposition 4.1. *Let A be a Krein-selfadjoint operator on a Krein space V. Let $\mathfrak{J}$ be a fundamental symmetry, such that* $\mathrm{dom}(A) \cap \mathfrak{J}\mathrm{dom}(A)$ *is dense in V. Let $\langle \cdot, \cdot \rangle$ be the scalar product associated with $\mathfrak{J}$. Then the quadratic form*

$$q(\phi_1, \phi_2) := \tfrac{1}{2}(\langle A\phi_1, A\phi_2 \rangle + \langle A^*\phi_1, A^*\phi_2 \rangle) \tag{16}$$

on $\mathrm{dom}(A) \cap \mathfrak{J}\mathrm{dom}(A)$ *is closed and the unique selfadjoint operator* $(A)_{\mathfrak{J}}$ *associated with this form commutes with $\mathfrak{J}$. Therefore, it is Krein-selfadjoint. Moreover* $\mathrm{dom}((A)_{\mathfrak{J}}^{1/2}) = \mathrm{dom}(A) \cap \mathfrak{J}\mathrm{dom}(A)$.

Proof. Since both A and A^* are closed, the quadratic form is closed as well. We repeat the construction of the selfadjoint operator associated with this form (see [19, Theorem VIII.15]). Denote $W = \mathrm{dom}(A) \cap \mathfrak{J}\mathrm{dom}(A)$. The pairing of the scalar product yields an inclusion of spaces $W \subset V \subset W^*$, where W^* is the dual space of W. We define the operator $\hat{B} : W \to W^*$ by $[\hat{B}\phi](\psi) := q(\psi, \phi) + \langle \psi, \phi \rangle$. $\hat{B}$ is in isometric isomorphism. With $\mathrm{dom}(B) := \{\psi \in W; \hat{B}\psi \in V\}$ the operator $B := \hat{B}|_{\mathrm{dom}(B)} : \mathrm{dom}(B) \to V$ is selfadjoint and $(A)_{\mathfrak{J}} = B - 1$. By construction $\mathfrak{J}$ restricts on W to a norm preserving isomorphism and its adjoint map $\hat{\mathfrak{J}}^* : W^* \to W^*$ is the continuous extension of $\mathfrak{J}$. From the definition of $\hat{B}$ we get immediately $\hat{B}\mathfrak{J} = \hat{\mathfrak{J}}^*\hat{B}$. Therefore, $\mathrm{dom}((A)_{\mathfrak{J}}) = \mathrm{dom}(B)$ is invariant under the action of $\mathfrak{J}$ and furthermore $(A)_{\mathfrak{J}}$ and $\mathfrak{J}$ commute. The form domain of $(A)_{\mathfrak{J}}$ is $\mathrm{dom}((A)_{\mathfrak{J}}^{1/2})$, and we conclude that $\mathrm{dom}((A)_{\mathfrak{J}}^{1/2}) = W$.

We therefore make the following definition.

Definition 4.2. Let A be a Krein-selfadjoint operator on a Krein space V and suppose that $\mathfrak{J}$ is a fundamental symmetry such that $\mathrm{dom}(A) \cap \mathfrak{J}\mathrm{dom}(A)$ is dense in V. Then the $\mathfrak{J}$-modulus $[A]_{\mathfrak{J}}$ of A is the Krein-selfadjoint operator $(A)_{\mathfrak{J}}^{1/2}$ constructed above.

4.3. Ideals of operators on Krein spaces

Since all $\mathfrak{J}$-inner products define equivalent norms, properties of operators like boundedness and compactness, which depend only on the topological structure of the Hilbert space, carry over to Krein spaces without change. The algebra of bounded operators in a Krein space V will be denoted by $\mathcal{B}(V)$. Each fundamental symmetry $\mathfrak{J}$ defines a norm on $\mathcal{B}(V)$ by $\|a\|_{\mathfrak{J}} := \sup_v(\|av\|_{\mathfrak{J}}/\|v\|_{\mathfrak{J}})$, where $\|v\|_{\mathfrak{J}}^2 = (v, \mathfrak{J}v)$. The norms on $\mathcal{B}(V)$ induced by different fundamental symmetries are equivalent. We choose a fundamental symmetry $\mathfrak{J}$ and view V as a Hilbert space with the $\mathfrak{J}$-inner product. Since $\mathcal{L}^{p+}$ are ideals in $\mathcal{B}(V)$, we have $B^{-1}\mathcal{L}^{p+}B = \mathcal{L}^{p+}$ for any invertible operator in $\mathcal{B}(V)$. Therefore, the definition of $\mathcal{L}^{p+}$ does not depend on the choice of scalar product and consequently it is independent of the chosen fundamental symmetry. The same argument applies to the Dixmier traces. Let $\omega \in \Gamma_s(l^\infty)$ be fixed. Then for any $a \in \mathcal{L}^{1+}$ and any invertible operator in $\mathcal{B}(V)$ we have $\mathrm{Tr}_\omega(B^{-1}aB) = \mathrm{Tr}_\omega(a)$. Therefore, the Dixmier trace does not depend on the choice of fundamental symmetry. We conclude that both $\mathcal{L}^{p+}$ and Tr_ω make sense on Krein spaces without referring to a particular fundamental symmetry.

5. Clifford algebras and the Dirac operator

5.1. Clifford algebras and the spinor modules

Let $q_{n,k}$ be the quadratic form $q(x) = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$ on $\mathbb{R}^n$. The Clifford algebra $\mathrm{Cl}_{n,k}$ is the algebra generated by the symbols $c(x)$ with $x \in \mathbb{R}^n$ and the relations

$$x \to c(x) \text{ is linear,} \tag{17}$$

$$c(x)^2 = q_{n,k}(x)1. \tag{18}$$

Let $\mathrm{Cl}_{n,k}^c$ be the complexification of $\mathrm{Cl}_{n,k}$ endowed with the antilinear involution $^+$ defined by $c(v)^+ = (-1)^k c(v)$. For n even the algebra $\mathrm{Cl}_{n,k}^c$ is isomorphic to the matrix algebra $\mathrm{Mat}_{\mathbb{C}}(2^{n/2})$, for n odd it is isomorphic to $\mathrm{Mat}_{\mathbb{C}}(2^{[n/2]}) \oplus \mathrm{Mat}_{\mathbb{C}}(2^{[n/2]})$. Let $\sigma_1, \sigma_2, \sigma_3$ be the Pauli matrices and define

$$\tau(j) = \begin{cases} \mathrm{i} & \text{for } j \leq k, \\ 1 & \text{for } j > k. \end{cases}$$

For n even we define the isomorphism $\Phi_{n,k} : \mathrm{Cl}^c_{n,k} \to \mathrm{Mat}_{\mathbb{C}}(2^{\frac{n}{2}})$ by

$$\Phi_{n,k}(c(x_{2j+1})) := \tau(2j+1) \cdot \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{j\text{-times}} \otimes \sigma_1 \otimes 1 \otimes \cdots \otimes 1,$$

$$\Phi_{n,k}(c(x_{2j})) := \tau(2j) \cdot \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{(j-1)\text{-times}} \otimes \sigma_2 \otimes 1 \otimes \cdots \otimes 1.$$

Whereas for odd $n = 2m + 1$ we define $\Phi_{n,k} : \mathrm{Cl}^c_{n,k} \to \mathrm{Mat}_{\mathbb{C}}(2^{[n/2]}) \oplus \mathrm{Mat}_{\mathbb{C}}(2^{[n/2]})$ by

$$\Phi_{n,k}(c(x_j)) := \begin{cases} \Phi_{2m,k}(c(x_j)) \oplus \Phi_{2m,k}(c(x_j)) & \text{for } 1 \leq j \leq 2m, \\ \tau(j)\{(\sigma_3 \otimes \cdots \otimes \sigma_3) \oplus (-\sigma_3 \otimes \cdots \otimes \sigma_3)\} & \text{for } j = 2m+1. \end{cases}$$

For even n the isomorphism $\Phi_{n,k}$ gives an irreducible representation of $\mathrm{Cl}^c_{n,k}$ on $\Delta_{n,k} := \mathbb{C}^{2^{n/2}}$, whereas for n odd we obtain an irreducible representation on $\Delta_{n,k} := \mathbb{C}^{2^{[n/2]}}$ by restricting $\Phi_{n,k}$ to the first component. The restrictions of these representations to the group $\mathrm{Spin}(n, k) \subset \mathrm{Cl}_{n,k}$ are the well known spinor representations on $\Delta_{n,k}$. In the following we write $\gamma(v)$ for the image of $c(v)$ under this representation. In case n is even we define the grading operator $\chi := \mathrm{i}^{(n(n-1)/2)+k}\gamma(x_1) \cdots \gamma(x_n)$. We have

$$\chi^2 = 1, \tag{19}$$

$$\chi\gamma(v) + \gamma(v)\chi = 0. \tag{20}$$

There is no analogue to this operator in the odd-dimensional case. There is a natural non-degenerate indefinite inner product on the modules $\Delta_{n,k}$ given by

$$(u, v) = \mathrm{i}^{(k(k+1)/2)}\langle\gamma(x_1) \cdots \gamma(x_k)u, v\rangle_{\mathbb{C}^{2^{[n/2]}}}. \tag{21}$$

This indefinite inner product is invariant under the action of the group $\mathrm{Spin}(n, k)^+$ which is the double covering group of $\mathrm{SO}(n, k)^+$. Furthermore the Krein-adjoint $\Phi_{n,k}(x)^+$ of $\Phi_{n,k}(x)$ is given by $\Phi_{n,k}(x^+)$. If n is even, one gets for the grading operator $\chi^+ = (-1)^k\chi$. Up to a factor this inner product is uniquely determined by these properties.

5.2. Fundamental symmetries of the spinor modules

Let now n and k be fixed and denote by g the unique bilinear form on $\mathbb{R}^n$ such that $g(v, v) = q_{n,k}(v)$. A spacelike reflection is linear map $r : \mathbb{R}^n \to \mathbb{R}^n$ with $r^2 = 1$, $g(ru, rv) = g(u, v)$ for all $u, v \in \mathbb{R}^n$ such that $g(\cdot, r\cdot)$ is a positive definite inner product. Each such reflection determines a splitting $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$ into g-orthogonal eigenspaces of r for eigenvalue -1 and $+1$. Clearly, g is negative definite on the first and positive definite on the second summand. Conversely for each splitting of $\mathbb{R}^n$ into a direct sum of g-orthogonal subspaces such that g is negative or positive definite on the summands determines a spacelike reflection.

To each such spacelike reflection we can associate a fundamental symmetry of the Krein space $\Delta_{n,k}$. We choose an oriented orthonormal basis $(e_1, \ldots, e_k)$ in the eigenspace for eigenvalue -1. Then the operator $\mathfrak{J}_r := \mathrm{i}^{(k(k+1)/2)}\gamma(e_1)\cdots\gamma(e_k)$ is a fundamental symmetry of $\Delta_{n,k}$ and we have $\mathfrak{J}_r\gamma(v)\mathfrak{J}_r = (-1)^k\gamma(rv)$. In general, not all the fundamental symmetries are of this form. The following criterion will turn out to be useful.

Proposition 5.1. *Let $\mathfrak{J}$ be a fundamental symmetry of the Krein space $\Delta_{n,k}$ such that for each $v \in \mathbb{R}^n \subset \mathrm{Cl}^c_{n,k}$ the matrix*

$$(\mathfrak{J}\gamma(v))^2 + (\gamma(v)\mathfrak{J})^2$$

is proportional to the identity. If n is even assume furthermore that $\mathfrak{J}$ commutes or anticommutes with the grading operator χ. Then there is a spacelike reflection r such that $\mathfrak{J} = \mathfrak{J}_r$.

Proof. By assumption $h(u, v) = (-1)^k\{\mathfrak{J}\gamma(u)\mathfrak{J}, \gamma(v)\} = \{\gamma(u)^*, \gamma(v)\}$ is a real valued bilinear form on $\mathbb{R}^n$, where $\{\cdot, \cdot\}$ denotes the anti-commutator and the * is the adjoint in the $\mathfrak{J}$-scalar product. Clearly, $h(v, v) \geq 0$ for all $v \in \mathbb{R}^n$. Therefore, there exists a matrix $a \in \mathrm{End}(\mathbb{R}^n)$ such that

$$\{\gamma(u)^*, \gamma(v)\} = \{\gamma(au), \gamma(v)\} \tag{22}$$

for all $u, v \in \mathbb{R}^n$. As a consequence $\delta(u) = \gamma(u)^* - \gamma(au)$ anti-commutes with all elements $\gamma(v)$. In the odd-dimensional case there is no such matrix other than 0 and in the even-dimensional case δ must be a multiple of the grading operator. Therefore, $\gamma^*(v) = (-1)^k\mathfrak{J}\gamma(v)\mathfrak{J}$ can be written as a sum $\gamma(av) + \lambda(v)\chi$, where λ is a linear form on $\mathbb{R}^n$. From $\gamma(v)^{**} = \gamma(v)$ and $\mathfrak{J}\chi\mathfrak{J} = \pm\chi$ we get $a^2 = 1$. For eigenvectors $av = \pm v$ of a one gets from the equation $(\gamma(v)^*)^2 = \gamma(v)^2$ that $\lambda(v)^2 = 0$. Hence, $\lambda = 0$. We showed that for n even or odd we always have $\mathfrak{J}\gamma(v)\mathfrak{J} = (-1)^k\gamma(av)$, for a reflection a. The bilinear form h is $h(u, v) = (u, av)$ and since it is positive semi-definite and a has trivial kernel, it is positive definite. Therefore, a is a spacelike reflection and consequently $\mathfrak{J}_a\gamma(v)\mathfrak{J}_a = (-1)^k\gamma(av)$. It remains to show that $\mathfrak{J}_a = \mathfrak{J}$. From the above relation one gets $\mathfrak{J}\mathfrak{J}_a\gamma(v)\mathfrak{J}_a\mathfrak{J} = \gamma(v)$, and therefore, $\mathfrak{J}\mathfrak{J}_a$ commutes with all $\gamma(v)$ and has to be a multiple of the identity. Hence, $\mathfrak{J} = z\mathfrak{J}_a$ for some complex number z. From $\mathfrak{J}^2 = \mathfrak{J}_a^2 = 1$, $\mathfrak{J}_a^+ = \mathfrak{J}_a$ and $\mathfrak{J}^+ = \mathfrak{J}$ we get $z = \pm 1$. Since the fundamental symmetries both give rise to positive definite scalar products on $\Delta_{n,k}$, we conclude that $z = 1$ and $\mathfrak{J} = \mathfrak{J}_a$.

5.3. Pseudo-Riemannian geometry and the Dirac operator

Let M be a smooth n-dimensional manifold. A pseudo-Riemannian metric g on M is a smooth section in the bundle $T^*M \otimes T^*M$, such that for all $x \in M$ the bilinear form g_x

on $T_x^*M \times T_x^*M$ is non-degenerate. If $g_x(v, v) = q_{n,k}(v)$ for a special choice of basis we say that g_x has signature (n, k). If g_x has signature (n, k) for all $x \in M$ the metric is called pseudo-Riemannian. If the signature is $(n, 0)$ then the metric is called Riemannian, in case the signature is $(n, 1)$ the metric is called Lorentzian. A vector field ξ is called timelike (spacelike, lightlike) if $g(\xi, \xi) < 0$, (>0, $=0$). The metric can be used to identify T^*M and TM and therefore, g can be regarded as a section in $TM \otimes TM$ inducing a scalar product on T_x^*M. See [1] or [2] for elementary properties of pseudo-Riemannian manifolds.

If (M, g) is a pseudo-Riemannian metric of signature (n, k), then the tangent-bundle TM can be split into an orthogonal direct sum $TM = F_1^k \oplus F_2^{n-k}$, where g is negative definite on F_1^k and positive definite on F_2^{n-k}. For such a splitting we can define a map $r : TM \to TM$ by $r(x, k_1 \oplus k_2) := (x, -k_1 \oplus k_2)$. Then the metric g^r defined by $g^r(a, b) := g(a, rb)$ is positive definite. Conversely suppose there is an endomorphism of vector bundles $r : TM \to TM$ with $g(r\cdot, r\cdot) = g$, $r^2 = \mathrm{id}$ and such that $g^r := g(\cdot, r\cdot)$ is positive definite. Then there is a splitting such that $r(x, k_1 \oplus k_2) = (x, -k_1 \oplus k_2)$. We call such maps spacelike reflections. Obviously, $r : TM \to TM$ is a spacelike reflection, if the restrictions of r to the fibres T_xM are spacelike reflections in the sense of the last section. In the following we call g^r the Riemannian metric associated with r.

In case the bundle TM, (F_1^k, F_2^{n-k}) is orientable the manifold is called orientable (time-orientable, space-orientable). Assume we are given an orientable, time-orientable pseudo-Riemannian manifold (M, g) of signature (n, k). Then the bundle of oriented orthonormal frames is an $\mathrm{SO}(n, k)^+$-principal bundle.

We saw that the metric information of a Riemannian manifold can be encoded in a spectral triple, where D was any Dirac type operator on some hermitian vector bundle E. In the case of pseudo-Riemannian manifolds there arises a major problem. Namely that Dirac type operators are not selfadjoint any more. We will see however that there exists a Krein space structure on the space of sections of E such that there are Krein-selfadjoint Dirac type operators. Assume now that M is an orientable time-orientable pseudo-Riemannian manifold. Let E be a vector bundle over M and assume that D is of Dirac type. This means that D is a first order differential operator and the principal symbol σ of D satisfies the relation

$$\sigma(\xi)^2 = g(\xi, \xi)\, \mathrm{id}_{E_p} \quad \forall \xi \in T_p^*M, \quad p \in M. \tag{23}$$

Therefore, $\gamma := \sigma$ satisfies the Clifford relations, which makes E a module for the Clifford algebra bundle. Let $r : TM \to TM$ be a spacelike reflection and identify TM with T^*M using the metric. Let $T^*M = F_1^k \oplus F_2^{n-k}$ be the splitting such that $r(x, k_1 \oplus k_2) = (x, -k_1 \oplus k_2)$. Then there is a hermitian structure $\langle \cdot, \cdot \rangle$ on E such that $\sigma_x(\xi)$ is anti-symmetric if $\xi \in F_1^k$ and symmetric if $\xi \in F_2^{n-k}$. Let $e_1, \ldots, e_k$ be a local oriented orthonormal frame for F_1^k and define $\mathfrak{J} := \mathrm{i}^{(k(k+1))/2} \gamma(e_1) \cdots \gamma(e_k)$. $\mathfrak{J}$ is independent of the choice of frames and the indefinite inner product $(\cdot, \cdot)_x := \langle \cdot, \mathfrak{J}(x) \cdot \rangle_x$ on E_x is non-degenerate. It makes E an non-degenerate indefinite inner product bundle. Moreover, $\mathrm{i}^k \sigma_x$ is symmetric with respect to this indefinite inner product. The space of square integrable sections of E is a Krein space endowed with the indefinite inner product structure

$$(f, g) := \int_M (f_x, g_x)_x \sqrt{|g|}\, \mathrm{d}x. \tag{24}$$

To each spacelike reflection r' we can associate a fundamental symmetry $\mathfrak{J}_{r'}$ of this Krein space by $\mathfrak{J}_{r'} := \mathrm{i}^{(k(k+1)/2)}\gamma(e_1)\cdots\gamma(e_k)$, where $e_1, \ldots, e_k$ is a local oriented orthonormal frame for F_1^k. We conclude that for a time-orientable orientable pseudo-Riemannian manifold there exists a Dirac type operator D on some non-degenerate indefinite inner product vector bundle E such that $\mathrm{i}^k D$ is symmetric with respect to this inner product. The following theorem was proved for Dirac operators on spin manifolds in [2]. For the sake of completeness and since the original proof is in german, we give a proof here.

Theorem 5.2 (Baum [2]). *Let E be a non-degenerate indefinite inner product vector bundle over an orientable time-orientable pseudo-Riemannian manifold $M^{n,k}$. Let $D : \Gamma_0(E) \to \Gamma_0(E)$ be a symmetric differential operator such that $\mathrm{i}^k D$ is of Dirac type. If there exists a spacelike reflection r such that the Riemannian metric associated with this reflection is complete, then D is essentially Krein-selfadjoint. In particular, if M is compact then D is always essentially Krein-selfadjoint.*

Proof. Let $\mathfrak{J}$ be the fundamental symmetry associated with the splitting and let $L^2(E)$ be the Hilbert space of sections which are square integrable with respect to the positive definite inner product induced by $\mathfrak{J}$. We denote this scalar product in the following by $\langle\cdot, \cdot\rangle$. It is clearly sufficient to show that $P = \mathfrak{J}D$ is essentially selfadjoint in $L^2(E)$. Note that P is a first order differential operator which is symmetric in $L^2(E)$. Therefore, it is closeable. The proof consists of two steps. Let $\mathrm{dom}_0(P^*)$ be the intersection of $\mathrm{dom}(P^*)$ with the space of compactly supported square integrable section. We first show that $\mathrm{dom}_0(P^*) \subset \mathrm{dom}(\overline{P})$. In the second step we show that $\mathrm{dom}_0(P^*)$ is dense in the Hilbert space $\mathrm{dom}(P^*)$ endowed with the scalar product $\langle x, y\rangle_{P^*} := \langle x, y\rangle + \langle P^*x, P^*y\rangle$. The combination of these results shows that $\mathrm{dom}(\overline{P})$ is dense in the Hilbert space $\mathrm{dom}(P^*)$, and therefore, P is essentially selfadjoint.

First step. Note that since P is symmetric, the operator P^* is a closed extension of P, and furthermore the adjoint operator $P' : \mathcal{D}'(E) \to \mathcal{D}'(E)$ is the continuous extension of P and P^* to the space of distributions. Assume that f is in $\mathrm{dom}_0(P^*)$. Then both f and $g = P^* f$ have compact support. Clearly, f is a weak solution to the equation $Pf = g$, hence, it is also a strong solution (see e.g. [21, Prop. 7.4]), i.e. there is a sequence f_n converging to f in the L^2-sense such that Pf_n converges to g also in the L^2-sense. Therefore, f is in $\mathrm{dom}(\overline{P})$.

Second step. Assume that $f \in \mathrm{dom}(P^*)$. We will construct a sequence f_n in $\mathrm{dom}_0(P^*)$ such that $f_n \to f$ and $P^* f_n \to P^* f$ in the L^2-sense. Fix an $x_0 \in M$ and let $\mathrm{dist}(x)$ be a regularized distance function from x_0 in the complete Riemannian metric associated with the splitting. Choose a function $\chi \in C_0^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$, $\chi(t) = 0$ for $t \geq 2$, $\chi(t) = 1$ for $t \leq 1$, and $|\chi'| \leq 2$. We set $\chi_n(x) := \chi((1/n)\mathrm{dist}(x))$. By completeness of the manifold, all χ_n are compactly supported. We define the sequence $f_n := \chi_n f$ and clearly, $f_n \in \mathrm{dom}_0(P^*)$. Denoting by σ the principal symbol of D, we have $P' f_n = -\mathrm{i}\mathfrak{J}\sigma(d\chi_n)f + \chi_n P' f$. Clearly, $\chi_n P' f \to P' f$ in the L^2-sense. For the first summand we have the estimate

$$\|\mathfrak{J}\sigma(d\chi_n)f\|^2 \leq \int_{B_{2n}-B_n} \frac{4}{n^2}\|f\|^2, \tag{25}$$

where B_r denotes the metric ball with radius r centred at x_0. Since the right hand side vanishes in the limit $n \to \infty$, we conclude that $f_n \to f$ and $P' f_n \to P' f$ in the L^2-sense. Therefore, $\mathrm{dom}_0(P^*)$ is dense in the Hilbert space $\mathrm{dom}(P^*)$.

Example 5.3. A spin structure on a time-oriented oriented pseudo-Riemannian manifold $M^{n,k}$ is an $\mathrm{Spin}(n,k)^+$-principal bundle P over $M^{n,k}$ together with a smooth covering η from P onto the bundle Q of oriented orthonormal frames, such that the following diagram is commutative.

$$\begin{array}{ccccc} P \times \mathrm{Spin}(n,k)^+ & \longrightarrow & P & \longrightarrow & M^{n,k} \\ \downarrow_{\eta\times\lambda} & & \downarrow_{\eta} & & \| \\ Q \times \mathrm{SO}(n,k)^+ & \longrightarrow & Q & \longrightarrow & M^{n,k} \end{array} \tag{26}$$

Here λ denotes the covering map $\mathrm{Spin}(n,k)^+ \to \mathrm{SO}(n,k)^+$. The spinor bundle S associated with a Spin structure is the associated bundle $P \times_\pi \Delta_{n,k}$, where π denotes the representation of $\mathrm{Spin}(n,k)^+$ on $\Delta_{n,k}$. Let $\nabla : \Gamma(S) \to \Gamma(S) \otimes \Lambda^1$ be the Levi–Civita connection on the spinor bundle. The Dirac operator $D\!\!\!/$ is defined by $-\mathrm{i}\gamma \circ \nabla$, where γ denotes the action of covector fields on sections of the spinor bundle by Clifford multiplication. $D\!\!\!/$ is clearly of Dirac type and it was shown in [2] (see also [3]) that the space of square integrable sections of S is a Krein space such that $\mathrm{i}^k D\!\!\!/$ is symmetric.

6. Pseudo-Riemannian spectral triples

Definition 6.1. A pseudo-Riemannian spectral triple is a tuple $(\mathcal{A}, D, \mathcal{H})$, where $\mathcal{A}$ is a pre-C^*-algebra of bounded operators on a Krein space $\mathcal{H}$ such that $a^* = a^+$, and D is a Krein-selfadjoint operator on $\mathcal{H}$, such that the commutators $[D, a]$ are bounded for all $a \in \mathcal{A}$. A pseudo-Riemannian spectral triple is called even if there is a distinguished operator χ, anticommuting with D and commuting with $\mathcal{A}$ and with $\chi^2 = 1$ and $\chi^+ = \pm\chi$. If such an operator does not exist we say the spectral triple is odd and set by definition $\chi = 1$. We call χ the grading operator.

Here a pre-C^*-algebra is defined to be a normed $*$-algebra whose closure is a C^*-algebra. We will see later that it is natural to assume in addition the existence of a fundamental symmetry $\mathfrak{J}$ which commutes with all elements in $\mathcal{A}$. This endows $\mathcal{H}$ with the structure of a Hilbert space and $\mathcal{A}$ becomes a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. However, at the moment such an assumption is not necessary and we will therefore not assume this explicitly.

For a pseudo-Riemannian spectral triple we can repeat the construction of differential forms almost unchanged. Denote again the universal differential envelope of $\mathcal{A}$ by $(\Omega\mathcal{A}, d)$. Clearly, the map

$$\pi : \Omega\mathcal{A} \to \mathcal{B}(\mathcal{H}), \qquad \pi(a_0, a_1, \ldots, a_n) := a_0[D, a_1]\cdots[D, a_n], \quad a_j \in \mathcal{A}$$

is a representation of $\Omega\mathcal{A}$ on $\mathcal{H}$ such that $\pi(a^*) = \pi(a)^+$ for all $a \in \Omega\mathcal{A}$, where $^+$ denotes the Krein adjoint. We define the graded two sided ideal $j_0 := \oplus_n j_0^n$ by

$$j_0^n := \{\omega \in \Omega^n\mathcal{A}; \pi(\omega) = 0\} \tag{27}$$

and as in the case of spectral triples $j = j_0 + dj_0$ is a graded two-sided differential ideal. We define $\Omega_D\mathcal{A} := \Omega\mathcal{A}/j$. Clearly, $\Omega^n_D\mathcal{A} \simeq \pi(\Omega^n\mathcal{A})/\pi(dj_0 \cap \Omega^n\mathcal{A})$.

Example 6.2. Suppose that $M^{n,k}$ is a compact time-orientable orientable pseudo-Riemannian spin manifold with spinor bundle E. Let $\mathcal{H}$ be the Krein space of square integrable sections of E and let $D = \mathrm{i}^k \not{D}$, where $\not{D}$ is the Dirac operator. Then the triple $(C^\infty(M), D, \mathcal{H})$ is a pseudo-Riemannian spectral triple, and in the same way as this is done for the Riemannian case one shows that as a graded differential algebra $\Omega_D C^\infty(M)$ is canonically isomorphic to the algebra of differential forms on M. We call this triple the canonical triple associated with M.

For Riemannian spin manifolds the differential structure is encoded in the Dirac operator. For example the space of smooth sections of the spinor bundle coincides with the space $\bigcap_n \mathrm{dom}(D^n)$. This is essentially due to the ellipticity of the Dirac operator. In the pseudo-Riemannian case the Dirac operator is not elliptic any more and sections in $\bigcap_n \mathrm{dom}(D^n)$ may be singular in the lightlike directions. We will circumvent this problem by introducing the notion of a smooth pseudo-Riemannian spectral triple. Let $(\mathcal{A}, D, \mathcal{H})$ be a pseudo-Riemannian spectral triple and suppose there is a fundamental symmetry, such that $\mathrm{dom}(D) \cap \mathfrak{J}\,\mathrm{dom}(D)$ is dense in $\mathcal{H}$. Then the operator $\Delta_{\mathfrak{J}} := ([D]^2_{\mathfrak{J}} + 1)^{1/2}$ is a self-adjoint operator on the Hilbert space $\mathcal{H}$ with scalar product $(\cdot, \mathfrak{J}\cdot)$. Let $\mathcal{H}^s_{\mathfrak{J}}$ be the closure of $\bigcap_n \mathrm{dom}(\Delta^n_{\mathfrak{J}})$ in the norm $\|\psi\|_s := \|\Delta^s_{\mathfrak{J}}\psi\|$. We have for $s > 0$ the equality $\mathcal{H}^s_{\mathfrak{J}} = \mathrm{dom}(\Delta^s_{\mathfrak{J}})$, and the indefinite inner product on $\mathcal{H}$ can be used to identify $\mathcal{H}^s_{\mathfrak{J}}$ with the topological dual of $\mathcal{H}^{-s}_{\mathfrak{J}}$. We define $\mathcal{H}^\infty_{\mathfrak{J}} := \bigcap_s \mathcal{H}^s_{\mathfrak{J}}$ and $\mathcal{H}^{-\infty}_{\mathfrak{J}} := \bigcup_s \mathcal{H}^s_{\mathfrak{J}}$. A map $a : \mathcal{H}^{-\infty} \to \mathcal{H}^{-\infty}$ is said to be in $\mathrm{op}^r_{\mathfrak{J}}$ if it continuously maps $\mathcal{H}^s$ to $\mathcal{H}^{s-r}$. Clearly, $\Delta_{\mathfrak{J}} \in \mathrm{op}^1_{\mathfrak{J}}$ and $\mathfrak{J} \in \mathrm{op}^0_{\mathfrak{J}}$. We introduce an equivalence relation on the set of fundamental symmetries $\mathfrak{J}$ such that $\mathrm{dom}(D) \cap \mathfrak{J}\,\mathrm{dom}(D)$ is dense in $\mathcal{H}$ in the following way. We say that $\mathfrak{J}_1 \sim \mathfrak{J}_2$ if $\mathcal{H}^s_{\mathfrak{J}_1} = \mathcal{H}^s_{\mathfrak{J}_2}$ as topological vector spaces. If we are dealing with a distinguished equivalence class, we will leave away the index $\mathfrak{J}$ and write e.g. op^r for $\mathrm{op}^r_{\mathfrak{J}}$ and $\mathcal{H}^s$ for $\mathcal{H}^s_{\mathfrak{J}}$, since these objects clearly depend only on the equivalence class of $\mathfrak{J}$. The spaces op^r have been introduced in [7] in the context of spectral triples.

Definition 6.3. A smooth pseudo-Riemannian spectral triple is defined to be a pseudo-Riemannian spectral triple $(\mathcal{A}, D, \mathcal{H})$ together with a distinguished non-empty equivalence class of fundamental symmetries $[\mathfrak{J}]$, such that $D \in \mathrm{op}^1$. We say a fundamental symmetry is smooth if $\mathfrak{J} \in [\mathfrak{J}]$.

Example 6.4. Suppose that $M^{n,k}$ is a compact orientable time-orientable pseudo-Riemannian spin manifold and let $(\mathcal{A}, D, \mathcal{H})$ be its canonical pseudo-Riemannian spectral triple. Let E be the spinor bundle. For each spacelike reflection r we constructed in the previous section a fundamental symmetry $\mathfrak{J}_r$ of the Krein space $\mathcal{H}$. The fundamental symmetries of the form $\mathfrak{J}_r$ belong to one and the same equivalence class and therefore define a smooth pseudo-Riemannian spectral triple. This can most easily be seen using the calculus of operators. Note that $\Delta_{\mathfrak{J}}$ is an elliptic classical operator of order 1. Therefore, $\mathcal{H}^s_{\mathfrak{J}}$ coincides with the Sobolev space $H_s(M, E)$ of sections of E and $\mathcal{H}^\infty$ coincides with the space of smooth sections $\Gamma(E)$. In the following we will think of the canonical triple associated with M as a smooth pseudo-Riemannian spectral triple with the above smooth structure.

Suppose that $(\mathcal{A}, D, \mathcal{H})$ is a smooth pseudo-Riemannian spectral triple. Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be smooth fundamental symmetries. Then $\Delta_{\mathfrak{J}_1}^{-1}$ is in op^{-1} and therefore, $\Delta_{\mathfrak{J}_2}\Delta_{\mathfrak{J}_1}^{-1}$ is bounded. As a consequence $\Delta_{\mathfrak{J}_1}^{-1}$ is in $\mathcal{L}^{p+}$ if and only if $\Delta_{\mathfrak{J}_1}^{-1}$ is in $\mathcal{L}^{p+}$.

Definition 6.5. We say a smooth pseudo-Riemannian spectral triple $(\mathcal{A}, D, \mathcal{H})$ is p^+-summable if for one (and hence for all) smooth fundamental symmetries $\mathfrak{J}$ the operator $\Delta_{\mathfrak{J}}^{-1}$ is in $\mathcal{L}^{p+}$.

For the canonical triple associated with a pseudo-Riemannian spin manifold we have a distinguished set of fundamental symmetries, namely those which are of the form $\mathfrak{J} = \mathfrak{J}_r$ for some spacelike reflection r. We may ask now if there is an analogue of this set in the general case.

Definition 6.6. Let $(\mathcal{A}, \mathcal{H}, D)$ be a smooth pseudo-Riemannian spectral triple. We say a fundamental symmetry $\mathfrak{J}$ is admissible if

1. $\mathfrak{J}$ is smooth.
2. $\mathfrak{J}\chi\mathfrak{J} = \chi^+$.
3. $\mathfrak{J}$ commutes with all elements of $\mathcal{A}$.
4. $\mathfrak{J}\pi(\Omega^p\mathcal{A})\mathfrak{J} = \pi(\Omega^p\mathcal{A})$.
5. $\mathfrak{J}\pi(dj_0 \cap \Omega^p\mathcal{A})\mathfrak{J} = \pi(dj_0 \cap \Omega^p\mathcal{A})$ if $p \geq 2$.

If $\mathfrak{J}$ is admissible and * denotes the adjoint in the $\mathfrak{J}$-inner product, then the above conditions imply that $a^* = a^+$ for all $a \in \mathcal{A}$, $\chi^* = \chi$ and that * leaves the spaces $\Omega_D^p\mathcal{A}$ invariant. The following theorem shows that in the case of a canonical triple associated with a spin manifold the set of admissible fundamental symmetries is canonically isomorphic to the set of spacelike reflections.

Theorem 6.7. *Suppose that $M^{n,k}$ is an orientable time-orientable compact pseudo-Riemannian spin manifold and let $(\mathcal{A}, D, \mathcal{H})$ be its canonical smooth triple with grading operator χ. Then the set of admissible fundamental symmetries coincides with the set*

$$\{\mathfrak{J}_r; r \text{ is a spacelike reflection}\}.$$

Proof. We first show that $\mathfrak{J}_r$ is admissible. Clearly, $\mathfrak{J}_r$ commutes with all elements of $\mathcal{A}$ and $\mathfrak{J}\chi\mathfrak{J} = \chi^+$. Moreover $\mathfrak{J}_r$ is smooth by construction (see Example 6.4). We need to show that the sets $\pi(\Omega^p\mathcal{A})$ and $\pi(dj_0 \cap \Omega^p\mathcal{A})$ are invariant under conjugation by $\mathfrak{J}_r$. We denote by γ the principal symbol of D. Since $[D, f] = -(\mathrm{i})^{k+1}\gamma(df)$, the space $\pi(\Omega^p\mathcal{A})$ is the set of operators of the form $\sum_j f^j\gamma(v_1^j)\cdots\gamma(v_p^j)$, where $f^j \in C^\infty(M)$ and $v_1^j, \ldots, v_p^j \in \Gamma(T^*M)$. Since $\mathfrak{J}_r\gamma(v)\mathfrak{J}_r = (-1)^k\gamma(rv)$, this space is invariant under conjugation by $\mathfrak{J}_r$. The proof of Proposition 7.2.2 in [16] shows that $\pi(dj_0 \cap \Omega^p\mathcal{A})$ coincides with the set of operators of the form $\sum_j f^j \gamma(v_1^j) \cdots \gamma(v_{p-2}^j)$, where $f^j \in \mathcal{A}$, $v_1^j, \ldots, v_k^j \in T^*M$. This set is also invariant under conjugation by $\mathfrak{J}_r$ and we conclude that $\mathfrak{J}_r$ is admissible. Suppose now we have another admissible fundamental symmetry $\mathfrak{J}$ in $[\mathfrak{J}_r]$. Since $\mathfrak{J} \in \mathrm{op}^0$, it acts continuously on $\Gamma(E)$ and since $\mathfrak{J}$ commutes with $\mathcal{A}$, it leaves the fibres invariant. It follows that $\mathfrak{J}$ is a smooth endomorphism of the spinor bundle. For a point $x \in M$ we denote by $\mathfrak{J}(x)$ the restriction of $\mathfrak{J}$ to the fibre at x. Since $\mathfrak{J}$ is admissible, $\mathfrak{J} \cdot \mathfrak{J}$ must leave the space of one-forms invariant. This implies that for all $v \in T_x^*M$ the matrix $\mathfrak{J}(x)\gamma(v)\mathfrak{J}(x)$ is again of the form $\gamma(u)$ for some $u \in T_x^*M$. By Proposition 5.1 there exists a spacelike reflection on the fibre at x inducing $\mathfrak{J}(x)$. Therefore, there is spacelike reflection r such that $\mathfrak{J} = \mathfrak{J}_r$.

Theorem 6.8. *Suppose that $M^{n,k}$ is a compact orientable time-orientable pseudo-Riemannian spin manifold and let $(\mathcal{A}, D, \mathcal{H})$ be the canonical smooth pseudo-Riemannian spectral triple associated with M. Then $(\mathcal{A}, D, \mathcal{H})$ is n^+-summable and for each $f \in C^\infty(M)$ and each admissible fundamental symmetry $\mathfrak{J}$ we have*

$$\int_M f = c(n)\,\mathrm{Tr}_\omega(f\Delta_{\mathfrak{J}}^{-n}), \tag{28}$$

where integration is taken with respect to the pseudo-Riemannian volume form $\sqrt{|g|}$ and $c(n) = 2^{n-[n/2]-1}\pi^{n/2} n\Gamma(n/2)$. Moreover with the same f and $\mathfrak{J}$

$$\mathrm{Tr}_\omega(fD^2\Delta_{\mathfrak{J}}^{-n-2}) = (-1)^k \frac{n-2k}{n} \mathrm{Tr}_\omega(f\Delta_{\mathfrak{J}}^{-n}). \tag{29}$$

Proof. Let g^r be the Riemannian metric associated with a spacelike reflection. By construction the metric volume form of g^r coincides with the metric volume form of the pseudo-Riemannian metric. Now the principal symbol σ_1 of $\Delta_{\mathfrak{J}_r}$ is given by $\sigma_1(k) = \sqrt{g^r(k,k)}$ for covectors $k \in T^*M$. Connes' trace formula gives Eq. (28). What is left is to show that Eq. (29) holds. The operator $D^2\Delta_{\mathfrak{J}}^{-n-2}$ is a classical pseudodifferential operator of order $-n$ and its principal symbol σ_2 is given by $\sigma_2(k) = (-1)^k g(k,k) g^r(k,k)^{-n/2-1}$. Therefore, the principal symbol of $fD^2\Delta_{\mathfrak{J}}^{-n-2}$ is $f\sigma_2$. In order to calculate the relevant Dixmier trace we have to integrate this symbol over the cosphere bundle in some Riemannian metric. The result will be independent of the chosen Riemannian metric. In case $\mathfrak{J} = \mathfrak{J}_r$ we use g^r to integrate. On the cosphere bundle σ_2 restricts to $(-1)^k g(k,k)$. Therefore,

$$\mathrm{Tr}_\omega(fD^2\Delta_{\mathfrak{J}}^{-n-2}) = \frac{1}{c(n)}(-1)^k\,\mathrm{Vol}(S^{n-1})^{-1}\int_{S^*M} f\cdot g. \tag{30}$$

For local integration we can choose an oriented orthonormal frame $k_1, \ldots, k_n$ such that $g(k_i, k_i) = -1$ for $i = 1, \ldots, k$ and $g(k_i, k_i) = 1$ for $i = k+1, \ldots, n$. This shows that

$$\begin{aligned}\int_{S^*M} f\cdot g &= (-1)^k \left(\int_M f\right)\cdot \int_{S^{n-1}} (-\xi_1^2 - \cdots - \xi_k^2 + \xi_{k+1}^2 + \cdots + \xi_n) \\ &= (-1)^k\,\mathrm{Vol}(S^{n-1})\frac{n-2k}{n}\int_M f,\end{aligned} \tag{31}$$

which concludes the proof.

Eq. (29) shows that one can indeed recover the signature from the spectral data and that the notion of integration is independent of the chosen admissible fundamental symmetry.

The conditions for a fundamental symmetry to be admissible are in a sense minimal and it is not clear at this point that one does not need further conditions in order to get a sensible noncommutative geometry. For example one may require in addition that the set $\mathcal{A} \cup [D, \mathcal{A}]$ is contained in the domain of smoothness of the derivation $\delta_{\mathfrak{J}}(\cdot) = [\Delta_{\mathfrak{J}}, \cdot]$. This is clearly true for admissible fundamental symmetries in the case of a canonical spectral triple associated with a manifold. In the general case however we can not expect this to hold. We think it is also worth noting that for the classical situation there exist a number of equivalent definitions of admissibility. For example one has

Proposition 6.9. *Let $(\mathcal{A}, D, \mathcal{H})$ be as in Theorem 6.8. Assume that $\mathfrak{J}$ is a smooth fundamental symmetry that commutes with all elements of $\mathcal{A}$ and $\mathfrak{J}\chi\mathfrak{J} = \chi^+$. Then $\mathcal{A} \cup [D, \mathcal{A}]$ is contained in the domain of the derivation $\delta_{\mathfrak{J}}$ if and only if $\mathfrak{J}$ is admissible.*

Proof. By assumption $\mathfrak{J}$ is a smooth endomorphism of the spinor bundle. Let $\mathfrak{J}(x)$ be the restriction to the fibre at x. Denote by σ the principal symbol of D. The principal symbol A of the second order pseudodifferential operator $\Delta_{\mathfrak{J}}^2$ is given by $A_x(v) = \frac{1}{2}(\mathfrak{J}\sigma_x(v)\mathfrak{J}\sigma_x(v) + \sigma_x(v)\mathfrak{J}\sigma_x(v)\mathfrak{J})$ for $v \in T_x^*M$. The principal symbol of $\Delta_{\mathfrak{J}}$ is $A^{1/2}$. Assume now that $[\Delta_{\mathfrak{J}}, a]$ is bounded for all a in $\mathcal{A} \cup [D, \mathcal{A}]$. Then the principal symbol of the first order operator $[\Delta_{\mathfrak{J}}, a]$ must vanish. This implies that $A_x^{1/2}(v)$ commutes with all $\sigma_x(u)$; $u \in T_X^*M$. Since the Clifford action is irreducible, $A_x(v)$ is a multiple of the identity and by Proposition 5.1 we have $\mathfrak{J} = \mathfrak{J}_r$ for some spacelike reflection r.

7. The noncommutative tori

Definition 7.1. Let θ be a pre-symplectic form on $\mathbb{R}^n$. We denote by A_θ the unital C^*-algebra generated by symbols $u(y)$, $y \in \mathbb{Z}^n$ and relations

$$u(y)^* = u(y)^{-1}, \tag{32}$$

$$u(y_1)u(y_2) = \mathrm{e}^{\mathrm{i}\pi\theta(y_1,y_2)}u(y_1 + y_2). \tag{33}$$

Let $\mathcal{S}(\mathbb{Z}^n)$ be the Schwarz space over $\mathbb{Z}^n$, i.e. the space of functions on $\mathbb{Z}^n$ with

$$\sup_{y\in\mathbb{Z}^n} (1 + |y|^2)^p |a(y)|^2 < \infty \quad \forall p \in \mathbb{N}. \tag{34}$$

The rotation algebra $\mathcal{A}_\theta$ is defined by

$$\mathcal{A}_\theta := \left\{ a = \sum_{y\in\mathbb{Z}^n} a(y)u(y);\ a \in \mathcal{S}(\mathbb{Z}^n) \right\}. \tag{35}$$

It is well known that the linear functional $\tau : \mathcal{A}_\theta \to \mathbb{C}$ defined by

$$\tau\left(\sum_r a(y)u(y)\right) := a(0) \tag{36}$$

is a faithful tracial state over $\mathcal{A}_\theta$. In particular we have $\tau(a^*a) = \sum_{y\in\mathbb{Z}^n} |a(y)|^2$. Note that A_θ is generated by the elements $u_k := u(e_k)$, where e_k are the basis elements in $\mathbb{Z}^n$. They satisfy the relations

$$u_k^* = u_k^{-1}, \tag{37}$$

$$u_k u_i = \mathrm{e}^{2\pi \mathrm{i}\theta_{ik}} u_i u_k. \tag{38}$$

Definition 7.2. The basic derivations $\delta_1, \dots, \delta_n$ on $\mathcal{A}_\theta$ are defined by

$$\delta_j\left(\sum_{y\in\mathbb{Z}^n} a(y)u(y)\right) := 2\pi\mathrm{i} \sum_{y\in\mathbb{Z}^n} y_j a(y)u(y). \tag{39}$$

One checks easily that these are indeed derivations.

Let $\mathcal{H}_\tau$ be the GNS-Hilbert space of the state τ. Since τ is faithful, $\mathcal{H}_\tau$ coincides with the closure of $\mathcal{A}_\theta$ in the norm $\|a\|_\tau^2 = \tau(a^*a)$. The basic derivations extend to closed skew-adjoint operators on $\mathcal{H}_\tau$. Denote by $\mathbb{R}^{n,k}$ the vector space $\mathbb{R}^n$ endowed with the indefinite metric $q_{n,k}$ and let $\mathrm{Cl}^c_{n,k}$ be the corresponding Clifford algebra. Let $\Delta_{n,k}$ be the natural Clifford module for $\mathrm{Cl}^c_{n,k}$. Denote by $\gamma(v)$ the representation of $\mathbb{R}^n \subset \mathrm{Cl}^c_{n,k}$ on $\Delta_{n,k}$. We choose a basis $\{e_i\}$ in $\mathbb{R}^n$ such that the $\gamma_i := \gamma(e_i)$ satisfy $\gamma_i^2 = -1$ for $i = 1, \dots, k$ and $\gamma_i^2 = +1$ for $i = k+1, \dots, n$. We have the following proposition.

Proposition 7.3. *Let $\mathcal{H} = \mathcal{H}_\tau \otimes \Delta_{n,k}$ and let D be the closure of the operator*

$$D_0 := \mathrm{i}^{k-1}\left(\sum_{i=1}^n \gamma_i\delta_i\right) \tag{40}$$

on $\mathcal{H}$ with domain $\mathrm{dom}(D_0) = \mathcal{A}_\theta \otimes \Delta_{n,k}$. *Then $\mathcal{H}$ is a Krein space with the indefinite inner product defined by*

$$(\psi_1 \otimes v_1, \psi_2 \otimes v_2) := \langle\psi_1, \psi_2\rangle_{\mathcal{H}_\tau}(v_1, v_2)_{\Delta_{n,k}} \tag{41}$$

and $(\mathcal{A}_\theta, \mathcal{H}, D)$ is a pseudo-Riemannian spectral triple. If n is even the triple $(\mathcal{A}_\theta, \mathcal{H}, D)$ is even.

Proof. Since $\Delta_{n,k}$ is finite-dimensional and decomposable, each decomposition of $\Delta_{n,k} = V^+ \oplus V^-$ into positive and negative definite subspaces gives rise to a decomposition $\mathcal{H} = \mathcal{H}_\tau \otimes V^+ \oplus \mathcal{H}_\tau \otimes V^-$. Clearly, the subspaces are intrinsically complete. Therefore, $\mathcal{H}$ is a Krein space. Next we show that D_0 is essentially Krein-selfadjoint on $\mathcal{H}$. Clearly, $\mathfrak{J} := \mathrm{i}^{(k(k+1)/2)}\gamma_1 \cdots \gamma_k$ is a fundamental symmetry of $\mathcal{H}$ and it is enough to show that the symmetric operator $\mathfrak{J}D_0$ is essentially selfadjoint on $\mathcal{H}$ endowed with the scalar product induced by $\mathfrak{J}$. The vectors $u(y) \in \mathcal{A}_\theta \subset \mathcal{H}_\tau$ form a total set in $\mathcal{H}_\tau$ and it is easy to see that the vectors of the form $u(y) \otimes \psi$ are analytic for $\mathfrak{J}D_0$. By Nelsons theorem $\mathfrak{J}D_0$ is essentially selfadjoint on $\mathcal{A}_\theta \otimes \Delta_{n,k} \subset \mathcal{H}$ and therefore, D_0 is essentially Krein-selfadjoint. For all $a \in \mathcal{A}_\theta$ we have $[D, a] = \mathrm{i}^{k-1} \sum_i \gamma_i(\delta_i a)$, which is clearly a bounded operator. Hence, $(\mathcal{A}_\theta, \mathcal{H}, D)$ is a pseudo-Riemannian spectral triple. For even n this triple is even and the grading operator χ is just the grading operator in $\mathrm{Cl}^c_{n,k}$ acting on the second tensor factor. In case n is odd the triple is odd and we set $\chi = 1$.

In the following we will need the image of the universal differential forms and the junk forms under the representation $\pi : \Omega\mathcal{A}_\theta \to \mathcal{B}(\mathcal{H})$ associated with $(\mathcal{A}_\theta, \mathcal{H}, D)$.

Lemma 7.4. *For the above defined pseudo-Riemannian spectral triple we have*

$$\pi(\Omega^m \mathcal{A}_\theta) = \left\{ \sum_j a^j \gamma(v_1^j) \cdots \gamma(v_m^j);\ a^j \in \mathcal{A}_\theta, v_i^j \in \Delta_{n,k} \right\}, \tag{42}$$

$$\pi(dj_0 \cap \Omega^m \mathcal{A}_\theta) = \left\{ \sum_j a^j \gamma(v_1^j) \cdots \gamma(v_{m-2}^j);\ a^j \in \mathcal{A}_\theta, v_i^j \in \Delta_{n,k} \right\}. \tag{43}$$

Proof. The first equation follows from the relation $[D, a] = \mathrm{i}^{k-1} \sum_i \gamma_i \delta_i(a)$. It remains to show that the second equation holds. Let ω be the $(m-1)$-form $(f_0 df_0 - df_0 f_0) df_1 \cdots df_{m-2}$ with $f_0 = u_l$. We have $\delta_i f_0 = 2\pi \mathrm{i} \delta_{il} f_0$. A short calculation shows that $\pi(\omega) = 0$ and therefore, the form

$$\pi(d\omega) = -8\pi^2 \gamma_l^2 f_0^2 [D, f_1] \cdots [D, f_m] \tag{44}$$

is an element of $\pi(dj_0 \cap \Omega^m \mathcal{A}_\theta)$. The $\mathcal{A}_\theta$-module generated by this form is the set A^m of elements of the form $\sum_j a^j \gamma(v_1^j) \cdots \gamma(v_{m-2}^j)$ with $a^j \in \mathcal{A}_\theta$ and $v_1^j, \ldots, v_{k-2}^j \in \Delta_{n,k}$. Therefore, $A^m \subset \pi(dj_0 \cap \Omega^m \mathcal{A}_\theta)$. In case $m-1 \geq n$ this shows that $\pi(\Omega^m \mathcal{A}_\theta) = \pi(dj_0 \cap \Omega^m \mathcal{A}_\theta)$ and the above formula is a consequence of this. We treat the case $m \leq n$. Suppose that $\omega = \sum_j f_0^j df_1^j \cdots df_{m-1}^j$ and that

$$\pi(\omega) = \gamma_{\mu_1} \cdots \gamma_{\mu_{m-1}} \sum_j f_0^j \delta_{\mu_1} f_1^j \cdots \delta_{\mu_{m-1}} f_{m-1}^j = 0. \tag{45}$$

This implies that

$$\sum_j f_0^j \delta_{[\mu_1} f_1^j \cdots \delta_{\mu_{m-1}]} f_{m-1}^j = 0, \tag{46}$$

where the square bracket indicates the complete anti-symmetrization of the indices. If we apply δ_{μ_0} to the left of this equation and anti-symmetrize in all indices we obtain

$$\sum_j \delta_{[\mu_0} f_0^j \delta_{\mu_1} f_1^j \cdots \delta_{\mu_{m-1}]} f_{m-1}^j = 0. \tag{47}$$

Since

$$\pi(d\omega) = \gamma_{\mu_0} \cdots \gamma_{\mu_{m-1}} \sum_j \delta_{\mu_0} f_0^j \delta_{\mu_1} f_1^j \cdots \delta_{\mu_{m-1}} f_{m-1}^j, \tag{48}$$

we finally obtain $\pi(d\omega) \in A^m$.

The above lemma implies that $\Omega_D \mathcal{A}_\theta \cong \bigoplus_m \mathcal{A}_\theta \otimes \Lambda^m \mathbb{R}^n$ and the differential is given by $d(a \otimes e_1 \wedge \cdots \wedge e_k) = \sum_i \delta_i(a) \otimes e_i \wedge e_1 \wedge \ldots \wedge e_k$. Here e_i is a distinguished basis in $\mathbb{R}^n$.

Each spacelike reflection in $\mathbb{R}^{n,k}$ induces a fundamental symmetry $\tilde{\mathfrak{J}}_r$ of $\Delta_{n,k}$ and clearly, $\mathfrak{J}_r := \mathrm{id} \otimes \tilde{\mathfrak{J}}_r$ is a fundamental symmetry of $\mathcal{H}$. All these fundamental symmetries are in fact equivalent and hence induce the same smooth structure on $(\mathcal{A}_\theta, \mathcal{H}, D)$.

Proposition 7.5. *Let r_1 and r_2 be two spacelike reflections of $\mathbb{R}^{n,k}$. Then $\mathfrak{J}_{r_1} \sim \mathfrak{J}_{r_2}$, i.e.* $\mathrm{dom}(\Delta^s_{\mathfrak{J}_{r_1}})$ *and* $\mathrm{dom}(\Delta^s_{\mathfrak{J}_{r_2}})$ *are equal and carry the same topology for all $s \in \mathbb{R}$. Moreover* $\mathcal{H}^\infty = \mathcal{A}_\tau \otimes \Delta_{n,k}$.

Proof. Let E be the spinor bundle on the commutative torus $T^{n,k}$ with the flat pseudo-Riemannian metric of signature (n, k). Let $\mathcal{H}_c$ be the Krein space of square integrable sections of E. The map $W : \mathcal{H}_\tau \to L^2(T^{n,k})$ defined by $W(u(y)) = \mathrm{e}^{2\pi i (y,x)}$ is unitary and satisfies $W\delta_i W^{-1} = \partial_i$. The map $U := W \otimes \mathrm{id}$ is an isometric isomorphism of the Krein spaces $\mathcal{H}$ and $\mathcal{H}_c$ and UDU^{-1} coincides with $\mathrm{i}^k \not{D}$, where $\not{D}$ is the Dirac operator on the torus. Furthermore $U\mathfrak{J}_r U^{-1}$ are admissible fundamental symmetries of the canonical spectral triple associated with $T^{n,k}$. As a consequence the operators $U\Delta_{\mathfrak{J}_r} U^{-1}$ are classical pseudodifferential operators of first order and hence, $\mathrm{dom}(\Delta^s_{\mathfrak{J}_r}) = U^{-1} H_s(E)$ for $s > 0$ where $H_s(E)$ is the space of Sobolev sections of order s of E. Therefore, $\mathfrak{J}_{r_1} \sim \mathfrak{J}_{r_2}$. The equation $\mathcal{H}^\infty = \mathcal{A}_\tau \otimes \Delta_{n,k}$ follows from $W^{-1} C^\infty(T^{n,k}) = \mathcal{A}_\theta \subset \mathcal{H}_\tau$, which is easy to check.

We view in the following $(\mathcal{A}_\theta, \mathcal{H}, D)$ as a smooth pseudo-Riemannian spectral triple with the above defined smooth structure and refer to it as the noncommutative pseudo-Riemannian torus $T^{n,k}_\theta$. For simplicity we restrict our considerations to the case where the algebra $\mathcal{A}_\theta$ has trivial center.

Theorem 7.6. *Suppose that $\mathcal{A}_\theta$ has trivial center. Then the set of admissible fundamental symmetries of $(\mathcal{A}_\theta, \mathcal{H}, D)$ coincides with the set*

$$\{\mathfrak{J}_r; r \text{ is a spacelike reflection of } \Delta_{n,k}\}.$$

Proof. Let r be a spacelike reflection of $\Delta_{n,k}$. By construction $\mathfrak{J}_r$ is smooth. We first show that $\mathfrak{J}_r$ is admissible. Clearly, $\mathfrak{J}_r$ commutes with all elements of $\mathcal{A}_\theta$ and $\mathfrak{J}\chi\mathfrak{J} = \chi^+$. Lemma 7.4 shows that indeed $\mathfrak{J}\pi(j \cap \Omega^p \mathcal{A})\mathfrak{J} = \pi(j \cap \Omega^p \mathcal{A})$ and $\mathfrak{J}\pi(\Omega\mathcal{A})\mathfrak{J} = \pi(\Omega\mathcal{A})$. Therefore, $\mathfrak{J}_r$ is admissible. Now suppose conversely that $\mathfrak{J}$ is an admissible fundamental symmetry. Since $\mathfrak{J}$ commutes with $\mathcal{A}_\theta$, we can view $\mathfrak{J}$ as an element in $\mathcal{A}'_\theta \otimes \mathrm{End}(\Delta_{n,k})$, where $\mathcal{A}'_\theta$ is the commutant of $\mathcal{A}_\theta$ in $\mathcal{B}(\mathcal{H}_\tau)$. Since $\mathfrak{J}$ is smooth, it is even an element of $\mathcal{A}^{\mathrm{opp}}_\theta \otimes \mathrm{End}(\Delta_{n,k})$, where $\mathcal{A}^{\mathrm{opp}}_\theta$ denotes the opposite algebra of $\mathcal{A}_\theta$ which acts on $\mathcal{H}_\tau$ from the right. The space $\pi(\Omega^1 \mathcal{A}_\theta)$ is invariant under conjugation by $\mathfrak{J}$. Therefore, the matrices $\mathfrak{J}\gamma_i\mathfrak{J}$ must commute with all elements of $\mathcal{A}_\theta^{\mathrm{opp}}$ and therefore have entries in the center of $\mathcal{A}_\theta^{\mathrm{opp}}$, which is trivial. Hence, the vector space spanned by the γ_i is invariant under conjugation by $\mathfrak{J}$. In the same way as in the proof of Proposition 5.1 one checks that the map $r : \mathbb{R}^{n,k} \to \mathbb{R}^{n,k}$ defined by $\mathfrak{J}\gamma(v)\mathfrak{J} = (-1)^k\gamma(rv)$ is a spacelike reflection. Hence, there exists a spacelike reflection r of $\Delta_{n,k}$ such that $\mathfrak{J}\gamma_i\mathfrak{J} = \mathfrak{J}_r\gamma_i\mathfrak{J}_r$. Denote by a the operator $\mathfrak{J}_r\mathfrak{J}$. Then a commutes with all γ_i and commutes with χ. Hence, $a \in \mathcal{A}_\theta^{\mathrm{opp}}$ and therefore, a commutes with $\mathfrak{J}_r$. We finally get from $a^+a = aa^+ = 1$ the equality $a^2 = 1$. Since both $\mathfrak{J}_r$ and $\mathfrak{J}$ give rise to positive scalar products, a must be positive in the $\mathfrak{J}_r$-scalar product and therefore, $a = 1$. We conclude that $\mathfrak{J} = \mathfrak{J}_r$.

Theorem 7.7. *Suppose that $\mathcal{A}_\theta$ has trivial center. The smooth pseudo-Riemannian spectral triple $(\mathcal{A}_\theta, \mathcal{H}, D)$ is n^+-summable and for all $a \in \mathcal{A}_\theta$ and each admissible fundamental symmetry $\mathfrak{J}$ we have*

$$\mathrm{Tr}_\omega(a\Delta_{\mathfrak{J}}^{-n}) = \frac{1}{c(n)}\tau(a), \tag{49}$$

$$\mathrm{Tr}_\omega(aD^2\Delta_{\mathfrak{J}}^{-n-2}) = (-1)^k\frac{n-2k}{n}\mathrm{Tr}_\omega(a\Delta_{\mathfrak{J}}^{-n}). \tag{50}$$

Proof. Let E be the spinor bundle on the commutative torus $T^{n,k}$ with the flat pseudo-Riemannian metric of signature (n, k). Let $\mathcal{H}_c$ be the Krein space of square integrable sections of E. In the proof of Proposition 7.5 we constructed an isomorphism of Krein spaces $U : \mathcal{H} \to \mathcal{H}_c$ such that $UDU^{-1} = \mathrm{i}^k\not{D}$, where $\not{D}$ is the Dirac operator on $T^{n,k}$. Moreover the $U\mathfrak{J}_rU^{-1}$ are admissible fundamental symmetries of the canonical spectral triple associated with $T^{n,k}$. Therefore, by Theorem 6.8 $\mathrm{Tr}_\omega(\Delta_{\mathfrak{J}}^{-n}) = c(n)^{-1}$ and $\mathrm{Tr}_\omega(D^2\Delta_{\mathfrak{J}}^{-n-2}) = (-1)^k\frac{n-2k}{n}\mathrm{Tr}_\omega(\Delta_{\mathfrak{J}}^{-n})$ for all admissible fundamental symmetries. The proof is finished if we can show that $\mathrm{Tr}_\omega(u(y)\Delta_{\mathfrak{J}}^{-n}) = 0$ and $\mathrm{Tr}_\omega(u(y)D^2\Delta_{\mathfrak{J}}^{-n-2}) = 0$ whenever $y \neq 0$. Let $\{\psi_i\}$ be an orthonormal basis in $\Delta_{n,k}$. Then the elements $\phi_{y,i} := u(y) \otimes \psi_i \in \mathcal{H}_\tau$ form an orthonormal basis in $\mathcal{H}$ and they are eigenvectors of $\Delta_{\mathfrak{J}}^{-n}$ and D^2. By Lemma 7.17 in [9] we have $\mathrm{Tr}_\omega(u(y)\Delta_{\mathfrak{J}}^{-n}) = \lim_{p\to\infty} \mathrm{Tr}(E_pu(y)\Delta_{\mathfrak{J}}^{-n})$, where E_p is the orthogonal projector onto the subspace generated by the first n eigenvectors of $\Delta_{\mathfrak{J}}^{-1}$ and whenever the limit exists. But since $\langle\phi_{y,i}, u(y')\phi_{y,i}\rangle_{\mathfrak{J}} = 0$ for all $y' \neq 0$, we get $\mathrm{Tr}(E_pu(y')\Delta_{\mathfrak{J}}^{-n}) = 0$ and consequently $\mathrm{Tr}_\omega(u(y')\Delta_{\mathfrak{J}}^{-n}) = 0$. The same argument gives $\mathrm{Tr}_\omega(u(y')D^2\Delta_{\mathfrak{J}}^{-n-2}) = 0$

8. Outlook

We showed that it is possible to extract the dimension, the signature and a notion of integration from the spectral data of a pseudo-Riemannian manifold. It would certainly be interesting if one could obtain the Einstein-Hilbert action in a similar way as in the Riemannian case (see [12,11]). This can probably not be done straightforwardly, but may require some averaging of expressions of the form $\mathrm{Wres}(D^2\Delta_{\mathfrak{J}}^{-n})$ over the set of admissible fundamental symmetries, where Wres denotes the Wodzicki residue.

Another interesting question is, which further conditions on the admissible fundamental symmetries are necessary in the general situation to guarantee that the functionals $\mathrm{Tr}_\omega(\cdot\,\Delta_{\mathfrak{J}}^{-n})$ and $\mathrm{Tr}_\omega(\cdot\,D^2\Delta_{\mathfrak{J}}^{-n-2})$ on the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$ do not depend on the choice of $\mathfrak{J}$.

As far as the noncommutative tori are concerned we believe that an analogue of Theorem 7.7 holds in case the center of $\mathcal{A}_\theta$ is not trivial. One should be able to proof this in a similar way as we did it here for the case of a trivial center.

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